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Laser heating of a two-layer system with constant surface absorption: an exact solution

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Abstract—Laser heating of a two-layer system is studied using the Laplace integral transform method. Expressions for the temperature profiles in the thin film and the substrate are obtained. The front surface temperature is also obtained. As an illustrative example the critical time required to initiate melting is computed for four systems. These systems are aluminum–glass, silver–glass, copper–glass and gold–glass. The obtained profiles in these systems are represented graphically. The obtained results reveal the linear dependence of the considered profiles on the heat power absorbed at the front surface. They also depend strongly on the thermal properties of the considered materials. Computations make it possible to decide whether one laser pulse can initiate melting in the considered target or not.

1. INTRODUCTION

Laser–solid interaction has aroused the considerable interest of many investigators [1–11].

This serious problem has different industrial applications [12–19]. Different models and techniques are used to obtain solutions for topics related to such problems.

Nevertheless, trials are still made in order to get solutions in a well-established form, applicable for practical computations. Recent studies have concentrated on the problem of laser heating and melting of a slab using the Fourier series expansion technique together with the differential and integral forms of the heat diffusion equation [20–24].

The present work aims to solve the laser heating problem for a two-layer system using the integral transform method. The main difficulty of such a technique is to find the inverse transform in order to recover the solution of the original problem. In the present trial an exact solution to the heat flow equation using the Laplace transform technique is derived. Expressions for the temperature profile in the thin film and substrate are obtained. The critical time required to initiate damage (melting) is computed from the obtained expression for the front surface temperature of the irradiated target.

The two-layer system is composed of a thin film of thickness d on a thick substrate. The two layers are in perfect thermal contact. Computations considering different materials constituting the two-layer system are made as illustrative examples.

2. THEORY

In setting up the problem, it is assumed that the incident laser irradiance q_0 on the front surface of the two-layer system is constant. It is partly reflected and partly absorbed. The absorbed energy flux at the front surface is simply $q_0 A_f$, where A_f is the absorptance of the thin film and is assumed to be temperature independent.

Two coincident x- and z-axes normal to the free surface of the considered system, along the direction of the incident irradiance, are used to describe the problem, where the boundary $x = 0$ represents the front surface of the thin film, while $z = 0$ represents the interface between the two layers, at $x = d$.

Loss arising from thermal radiation is neglected. The physical parameters of the thin film and the substrate are assumed to be temperature independent.

The heat flow is considered one-dimensional [1]. Moreover, it is suggested that no plasma is formed at the front surface for the considered values of the incident laser flux. The multireflections within the considered system are also neglected.

3. SOLUTIONS TO THE HEAT EQUATIONS

For the considered system, the heat diffusion equations for both the thin film and the substrate can be written as follows: for the thin film layer:

$$\frac{\partial T_f(x,t)}{\partial t} = \alpha_f \frac{\partial^2 T_f(x,t)}{\partial x^2}, \quad t > 0, \quad 0 \leqslant x \leqslant d; \quad (1)$$

for the substrate layer:

NOMENCLATURE

A_f optical surface absorptance of the thin film, dimensionless
B dimensionless parameter defined in the text
c_p specific heat [J kg^{-1} K^{-1}]
d thickness of thin film [m]
q_0 laser flux [W m^{-2}]
s Laplace transform variable
T excess temperature [K]
T_m melting temperature of material [K]
t time variable [s]
t_m critical time required to initiate melting [s]
t_s transit time, defined in the text [s]
x, z spatial variables [m].

Greek symbols
α thermal diffusivity [m^2 s^{-1}]
ε dimensionless parameter defined in the text
λ thermal conductivity [W m^{-1} K^{-1}]
λ' wavelength [microns]
ρ density [kg m^{-3}].

Subscripts
f related to the film layer
p related to the substrate layer.

$$\frac{\partial T_p(z,t)}{\partial t} = \alpha_p \frac{\partial^2 T_p(z,t)}{\partial z^2},$$

$$t > t_s, \quad 0 \leqslant z \leqslant \infty, \quad z = (x-d); \quad (2)$$

where T is the excess temperature with respect to the ambient temperature T_0, $\alpha = \lambda/\rho c_p$ is the thermal diffusivity in terms of the thermal conductivity λ, and the heat capacity per unit volume (ρc_p), t_s is the transit time, defined as the time taken for the excess temperature T of the rear surface of the thin film to change from zero.

The system of equations (1) and (2) is subjected to the following initial and boundary conditions:

$$T_f(x,0) = 0, \quad (3)$$

$$T_p(z,0) = 0. \quad (4)$$

The condition at the boundary $x = 0$ is:

$$-\lambda_f \frac{\partial T_f}{\partial x} = q_0 A_f. \quad (5)$$

The condition at the interface $x = d$ is:

$$T_f(d,t) = T_p(0,t), \quad (6)$$

$$-\lambda_f \frac{\partial T_f(d,t)}{\partial x} = -\lambda_p \frac{\partial T_p(0,t)}{\partial z}. \quad (7)$$

For the substrate one more condition is given as follows:

$$T_p(\infty,t) = 0. \quad (8)$$

Let us take the Laplace transform with respect to the time variable for both equations (1) and (2): considering the conditions (3) and (4), this gives:

$$\frac{d^2 \bar{T}_f(x,s)}{dx^2} - \frac{s}{\alpha_f} \bar{T}_f(x,s) = 0, \quad (9)$$

$$\frac{d^2 \bar{T}_p(z,s)}{dz^2} - \frac{s}{\alpha_p} \bar{T}_p(z,s) = 0, \quad (10)$$

where $\bar{T}_f(x,s)$ and $\bar{T}_p(z,s)$ denote the Laplace transform of T in the film and substrate regions, respectively. The Laplace transform of the considered conditions are as follows:

$$-\lambda_f \frac{\partial \bar{T}_f(0,s)}{\partial x} = \frac{q_0 A_f}{s}, \quad (11)$$

$$\bar{T}_f(d,s) = \bar{T}_p(0,s), \quad (12)$$

$$-\lambda_f \frac{\partial \bar{T}_f(d,s)}{\partial x} = -\lambda_p \frac{\partial \bar{T}_p(0,s)}{\partial z}, \quad (13)$$

$$\bar{T}_p(\infty,s) = 0. \quad (14)$$

The solutions of equations (9) and (10) can be written in the form:

$$\bar{T}_f(x,s) = c_1 \exp(+\sqrt{s/\alpha_f}\,x) + c_2 \exp(-\sqrt{s/\alpha_f}\,x), \quad (15)$$

$$\bar{T}_p(z,s) = c_3 \exp(+\sqrt{s/\alpha_p}\,z) + c_4 \exp(-\sqrt{s/\alpha_p}\,z). \quad (16)$$

Condition (14) gives $c_3 = 0$ and the solution in the substrate is written in the form:

$$\bar{T}_p(z,s) = c_4 \exp(-\sqrt{s/\alpha_p}\,z). \quad (17)$$

Condition (11) gives:

$$-\lambda_f \sqrt{s/\alpha_f}(c_1 - c_2) = \frac{q_0 A_f}{s}; \quad (18)$$

condition (12) gives:

$$c_4 = c_2 \exp(-\sqrt{s/\alpha_f}d) + c_1 \exp(+\sqrt{s/\alpha_f}d); \quad (19)$$

condition (13) gives:

$$c_4 \frac{\lambda_p}{\lambda_f} \frac{\sqrt{s/\alpha_p}}{\sqrt{s/\alpha_f}} = c_2 \exp(-\sqrt{s/\alpha_f}d) - c_1 \exp(+\sqrt{s/\alpha_f}d). \quad (20)$$

Equations (18)–(20) make it possible to get the unknown coefficients c_1, c_2 and c_4 in the following form:

$$c_1 = \frac{Bq_0 A_f e^{-2\sqrt{s/\alpha_f}d}}{\lambda_f s\sqrt{s/\alpha_f}[1 - B\exp(-2\sqrt{s/\alpha_f}d)]}, \quad (21)$$

$$c_2 = \frac{q_0 A_f}{\lambda_f s\sqrt{s/\alpha_f}[1 - B\exp(-2\sqrt{s/\alpha_f}d)]}, \quad (22)$$

$$c_4 = \frac{2q_0 A_f e^{-\sqrt{s/\alpha_f}d}}{\lambda_f s\sqrt{s/\alpha_f}(1+\varepsilon)[1 - B\exp(-2\sqrt{s/\alpha_f}d)]}, \quad (23)$$

where:

$$\varepsilon = \frac{\lambda_p}{\lambda_f}\frac{\sqrt{s/\alpha_p}}{\sqrt{s/\alpha_f}} = \frac{\lambda_p}{\lambda_f}\frac{\sqrt{\alpha_f}}{\sqrt{\alpha_p}} = \frac{\sqrt{\lambda_p \rho_p (c_p)_p}}{\sqrt{\lambda_f \rho_f (c_p)_f}}, \quad (24)$$

and

$$B = \frac{1-\varepsilon}{1+\varepsilon}, \quad B < 1. \quad (25)$$

Thus the solutions (15) and (17) can be written in the form:

$$\bar{T}_f(x,s) = \left\{\frac{q_0 A_f}{\lambda_f s\sqrt{s/\alpha_f}[1 - B\exp(-2\sqrt{s/\alpha_f}d)]}\right\} \times \{B\exp[-\sqrt{s/\alpha_f}(2d - x)] + \exp(-\sqrt{s/\alpha_f}x)\}, \quad (26)$$

and

$$\bar{T}_p(z,s) = \frac{2q_0 A_f \exp[-\sqrt{s/\alpha_f}(z\sqrt{\alpha_f/\alpha_p} + d)]}{\lambda_f s\sqrt{s/\alpha_f}(1+\varepsilon)[1 - B\exp(-2\sqrt{s/\alpha_f}d)]}. \quad (27)$$

Consider the following relations [25]:

$$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n, \quad |a| < 1. \quad (28)$$

This gives:

$$\frac{1}{[1 - B\exp(-2\sqrt{s/\alpha_f}d)]} = \sum_{n=0}^{\infty} [B^n \exp(-2n\sqrt{s/\alpha_f}d)]; \quad (29)$$

substituting equation (29) into equation (26) one gets:

$$\bar{T}_f(x,s) = \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f s\sqrt{s/\alpha_f}} \times \{B^{n+1}\exp(-\sqrt{s/\alpha_f}[2(n+1)d - x)] + B^n \exp[-\sqrt{s/\alpha_f}(2nd + x)]\}; \quad (30)$$

substituting equation (29) into equation (27) one gets:

$$\bar{T}_p(z,s) = \frac{2q_0 A_f}{\lambda_f s\sqrt{s/\alpha_f}(1+\varepsilon)} \sum_{n=0}^{\infty} \times B^n \exp\{-\sqrt{s/\alpha_f}[(1+2n)d + z\sqrt{\alpha_f/\alpha_p}]\}. \quad (31)$$

Let:

$$F(s) = \frac{1}{s\sqrt{s/\alpha_f}} \exp(-\sqrt{s/\alpha_f}x),$$

then the inverse transform of $F(s)$ is given by [26]:

$$L^{-1}\{F(s)\} = 2\sqrt{\alpha_f t/\pi}\exp\left(\frac{-x^2}{4\alpha_f t}\right) - x\,\mathrm{erfc}\frac{x}{\sqrt{4\alpha_f t}}, \quad x > 0, \quad \alpha_f > 0, \quad (32)$$

where erfc is the complementary error function. Substituting equation (32) into equations (30) and (31) one gets their inverse transformations in the form:

$$T_f(x,t) = \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1}\left(\sqrt{4\alpha_f t/\pi} \times \exp\left\{-\frac{[2d(1+n) - x]^2}{4\alpha_f t}\right\} - [2d(1+n) - x]\,\mathrm{erfc}\frac{2d(1+n) - x}{\sqrt{4\alpha_f t}}\right) + \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^n \left\{\sqrt{4\alpha_f t/\pi}\exp\left[-\frac{(2nd + x)^2}{4\alpha_f t}\right] - (2nd + x)\,\mathrm{erfc}\frac{(2nd + x)}{\sqrt{4\alpha_f t}}\right\}, \quad (33)$$

and

$$T_p(z,t) = \sum_{n=0}^{\infty} \frac{2q_0 A_f}{\lambda_f} \frac{B^n}{(1+\varepsilon)}\left(\sqrt{4\alpha_f t/\pi}\exp - \left\{\frac{[z\sqrt{\alpha_f/\alpha_p} + (1+2n)d]^2}{4\alpha_f t}\right\} - [z\sqrt{\alpha_f/\alpha_p} + (1+2n)d] \times \mathrm{erfc}\left\{\frac{[z\sqrt{\alpha_f/\alpha_p} + (1+2n)d]}{\sqrt{4\alpha_f t}}\right\}\right). \quad (34)$$

By substituting $x = d$ into equation (33) and $z = 0$ into equation (34) one can easily verify that condition (6) is fulfilled.

The temperature of the front surface is obtained by substituting $x = 0$ in equation (33). This gives:

$$T_f(0,t) = \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1}\left(\sqrt{4\alpha_f t/\pi} \times \exp\left\{-\frac{[2d(1+n)]^2}{4\alpha_f t}\right\}\right.$$

Table 1. The physical parameters [26, 27] for the chosen materials

Element	ρ (kg m^{-3})	λ (W m^{-1} K^{-1})	α (m^2 s^{-1})	c_p (J kg^{-1} K^{-1})	A_f	T_m (K)
Aluminum	2700	238	8.410×10^{-5}	0.896×10^3	0.056	633
Silver	10 524	418	17.00×10^{-5}	0.234×10^3	0.100	960
Copper	8954	386	11.25×10^{-5}	0.383×10^3	0.001	1056
Gold	19 320	315	6.500×10^{-5}	0.251×10^3	0.014	1037
Glass	2707	0.76	0.035×10^{-5}	0.800×10^3	—	—

$$-[2d(1+n)]\,\mathrm{erfc}\frac{2d(1+n)}{\sqrt{4\alpha_f t}}\Bigg)$$

$$+\sum_{n=0}^{\infty}\frac{q_0 A_f}{\lambda_f}B^n\left\{\sqrt{4\alpha_f t/\pi}\exp\left[-\frac{(2nd)^2}{4\alpha_f t}\right]\right.$$

$$\left.-2nd\,\mathrm{erfc}\frac{2nd}{\sqrt{4\alpha_f t}}\right\}. \tag{35}$$

The critical time t_m required to initiate melting (damage) at the front surface is computed. The transit time t_s can be estimated from equation (33) as the critical time required for the temperature at the interface $T_f(d, t_s)$ to start attaining values other than zero.

4. COMPUTATIONS AND DISCUSSION

As illustrative examples, computations for the following two-layer systems are carried out: aluminum–glass, copper–glass, silver–glass, and gold–glass. Each system is subjected to laser flux of density 10^{12} W m^{-2} and the metal thickness is 5 μ in all cases. The physical parameters for the chosen elements are obtained from refs. [26, 27] and are tabulated in Table 1.

In each case $T_f(x, t)$ and $T_p(z, t)$ are computed and represented graphically in Figs. 1–4. From such figures, it is clear that the temperature profile and the thermal depth in the substrate differ according to the material of deposited thin film at constant other operating conditions.

Moreover, the front surface temperature $T_f(0, t)$ in each case is obtained and shown graphically in Figs. 5 and 6. Such figures show that for silver and aluminum higher rates of heating than for gold and copper are recorded.

From Fig. 6, it is noticed that the rate of heating for either gold or copper seems to be approximately constant, so for practical purposes the dependence of $T_f(0, t)$ on the exposure time can be simplified through a linear relation.

$T_f(0, t)$ as computed from equation (25) shows the linear dependence of the front surface temperature $T_f(0, t)$ on the incident laser irradiance q_0, which is in agreement with other published articles [27, 28].

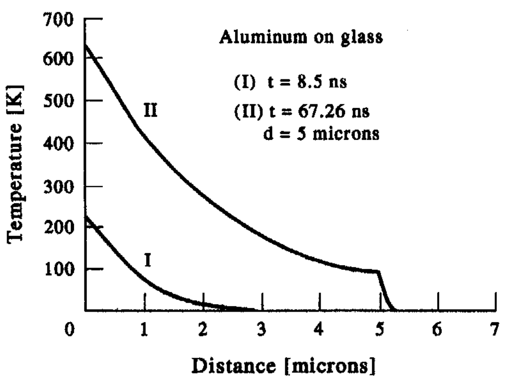

Fig. 1. Temperature profile within a two-layer system with constant surface absorptance (aluminum thin film deposited on glass substrate) at two different exposure times: (I) $t = 8.5$ ns; (II) $t = 67.26$ ns.

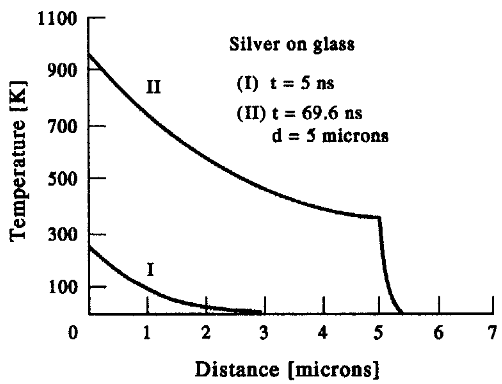

Fig. 2. Temperature profile within a two-layer system with constant surface absorptance (silver thin film deposited on glass substrate) at two different exposure times: (I) $t = 5$ ns; (II) $t = 69.6$ ns.

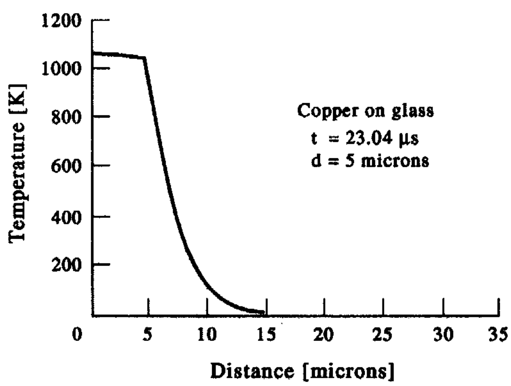

Fig. 3. Temperature profile within a two-layer system with constant surface absorptance (copper thin film deposited on glass substrate) at exposure time $t = 23.04$ μs.

Table 2. The critical time required to initiate damage

Element	q_0 (constant) $= 10^{12}$ W m^{-2}				q_0A_f (constant) $= 10^{10}$ W m^{-2}			
	q_0A_f	t_m (μs) present work	t_m (μs) $t < t_s$†	t_m (μs) $t > t_s$‡	q_0A_f	t_m (μs) present work	t_m (μs) $t < t_s$†	t_m (μs) $t > t_s$‡
Aluminum	56×10^9	0.06726	0.05737	—	10^{10}	0.8406	—	0.7966
Silver	10^{11}	0.06960	0.06315	—	10^{10}	1.2235	—	1.1309
Copper	10^9	23.040	—	18.034	10^{10}	1.8658	—	1.7368
Gold	14×10^9	1.744	—	1.667	10^{10}	2.5238	—	2.2845

† Equation (77) from ref. [20].
‡ Equation (82) from ref. [20].

The critical time required to initiate damage t_m is obtained for the four elements and is tabulated in Table 2, and is compared in the same table with the corresponding values computed according to other published equations [20] considering a thin film only.

From Table 2 the computations for t_m considering the constancy of either q_0 or the value of (q_0A_f) reveal that, among the four elements considered, gold or copper is more suitable to resist the damage induced by laser irradiance when small intervals of exposure time are considered: this result is of industrial importance.

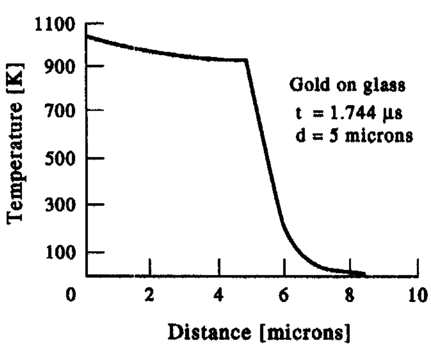

Fig. 4. Temperature profile within a two-layer system with constant surface absorptance (gold thin film deposited on glass substrate) at exposure time $t = 1.744$ μs.

The computations are carried out using double precision MS FORTRAN. Each of the series included in equations (33) and (34) is calculated separately where terms are added until the value of the added term is less than a specified tolerance. In our case the tolerance is taken to be 10^{-5}. The error function erf (x) is calculated using subroutine S15AEF of the (NAG PC 50) FORTRAN library and the complementary error function erfc x is then calculated since erfc $(x) = 1-\text{erf}\,(x)$.

5. CONCLUSIONS

In the light of the considered model and obtained computations for the problem of heating a two-layer system with constant surface absorptance the following conclusions are made:

1. The considered model and technique afford accurate expressions for the temperature profiles in the thin film and the substrate.
2. These two profiles are linear functions of the absorbed heat power (q_0A_f) at the irradiated front surface.
3. The critical time t_m required to initiate melting depends also on the value of the absorbed power (q_0A_f).
4. The considered profiles are no longer linear func-

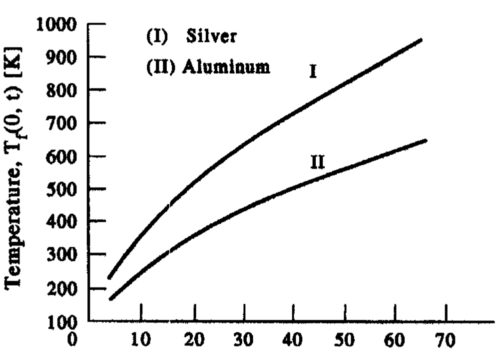

Fig. 5. The front surface temperature $T_f(0, t)$ against the exposure time t for two different two-layer systems with constant surface absorptance: (I) silver on glass; (II) aluminum on glass.

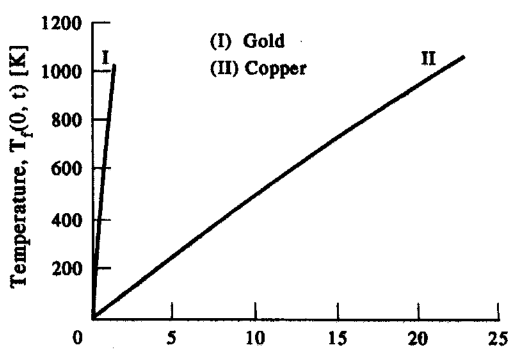

Fig. 6. The front surface temperature $T_f(0, t)$ against the exposure time t for two different two-layer systems with constant surface absorptance: (I) gold on glass; (II) copper on glass.

tions of the thermal properties of the materials constituting the two-layer system.

5. The rate of heating at the front surface also depends on the above-mentioned factors.
6. The computed results for t_m decide whether damage can be initiated within one laser pulse duration or not.

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